Semi-classical derivation of diamagnetism in atoms

While classical physics does not give rise to magnetism, it is surprizing what results can be obtained from simple semi-classical arguments. Here we derive the magnetic of an atom caused by the diamagnetic response to an external field. We see that we get the same result (8.20) as the proper quantum mechanical treatment.

Suppose that we have an electron moving in the xy-plane on a fixed circular orbit of radius r_{xy}, as in the Bohr model (admittedly, this is not very classical). Now we turn on a magnetic field in the z-direction. We can now apply Faraday's law

$$\oint E d\mathbf{l} = -\frac{\partial \Phi_B}{\partial t}. \tag{1}$$

to calculate the radial electric field acting on the electron. Faraday's law is equivalent to

$$E 2\pi r_{xy} = -\frac{d}{dt} B \pi r_{xy}^2, \tag{2}$$

and the electric field is thus

$$E = -\frac{r_{xy}}{2}\frac{d}{dt}B. \tag{3}$$

The electric field leads to a torque that accelerates the electron. The torque is $-eEr_{xy}$. Such a torque gives rise to a change in the angular momentum L of the electron and the rate of change is

$$\frac{d}{dt}L = \frac{er_{xy}^2}{2}\frac{d}{dt}B. \tag{4}$$

Starting from $B = 0$ and integrating this up in time to the full field B field gives the total change in angular momentum

$$\Delta L = \frac{er_{xy}^2}{2}B. \tag{5}$$

We can also write this as a change of the electron's speed on the circumference

$$\Delta L = \Delta v m_e r_{xy}, \tag{6}$$

and with this we obtain the change in the current

$$\Delta I = -e\frac{\Delta v}{2\pi r_{xy}} = -e\frac{\Delta L}{2\pi m_e r_{xy}^2}, \tag{7}$$

so that the induced magnetic moment is

$$\Delta\mu = \Delta I \pi r_{xy}^2 = -e\frac{\Delta L}{2m_e} = \frac{e^2 r_{xy}^2}{4m_e}B. \tag{8}$$

We now see that this is exactly the same as the quantum mechanical result (8.22) if we identify the expectation value of $x^2 + y^2$ with r_{xy}^2.