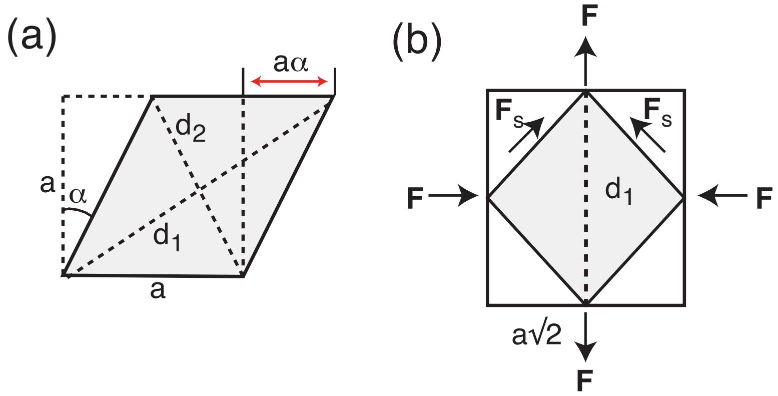

Figure 1: (a) Deformation of a cube of side length a by shear stress. (b) The same deformation brought about by the combination of combination of (conventional) tensile and compressive stress. For convenience, the strained cube is embedded in a bigger cuboid.

Relationship between elastic constants

The elastic constants describing a solid are closely related to each other. Here we prove the relation between the modulus of rigidity, Young's modulus and Poisson's ratio

$$G = \frac{Y}{2(1+\nu)}. \tag{1}$$

Another such relation is the subject of Problem 3.1.

To prove (1) we proceed in two steps. Consider first a cube of side length a that is exposed to shear stress. A side view of this cube is given in Fig. 1(a). We are interested in how much the length of the two diagonals d_1 and d_2 changes. To obtain this, we first determine the length of the distortion to the right (red arrow). This must be $a \tan\alpha \approx a\alpha$ for a small distortion. With this we calculate the diagonal length d_1 to be

$$d_1 = \sqrt{(a + a\alpha)^2 + a^2} \approx \sqrt{2a^2 + 2a^2\alpha}, \tag{2}$$

neglecting the quadratic term in α. We now expand d_1 for small α as

$$d_1 = \sqrt{2}a + \left.\frac{2a^2}{2\sqrt{2a^2 + 2a^2\alpha}}\right|_{\alpha=0} \alpha = \sqrt{2}a + \frac{1}{\sqrt{2}}a\alpha \tag{3}$$

The extension of d_1 is thus $a\alpha/\sqrt{2}$ and the strain along this direction is

$$\frac{\Delta d_1}{d_1} = \frac{\alpha a}{\sqrt{2}a\sqrt{2}} = \frac{\alpha}{2} = \frac{\tau}{2G}. \tag{4}$$

In the last step, we have used the definition of modulus of rigidity $G = \tau/\alpha$. Following exactly the same strategy for d_2, we find that this contracts by the same amount as d_1 expands.

We can now construct a situation in which we obtain the same deformation by combined application of conventional stress in two orthogonal directions. Since the deformation of the cube is symmetric (same expansion of d_1 as contraction of d_2) what we have to do is pull along the d_1 axis and push along the d_2 axis using the same force. In order to do this, we imagine the strained cube inside a bigger cuboid as shown in Fig. 1(b). The forces $\mathbf{F}$ acting on the sides all have the same magnitude. When we compare the force $\mathbf{F}$ to the forces $\mathbf{F}_S$ used to obtain the shear distortion, we see that $F = 2F_S/\sqrt{2} = \sqrt{2}F_S$. From this we obtain

$$\frac{F}{\sqrt{2}a^2} = \frac{\sqrt{2}F_S}{\sqrt{2}a^2} = \frac{F_S}{a^2}. \tag{5}$$

On the left hand side of this equation we recognize the magnitude of the stress σ needed to achieve the strained situation in Fig. 1(b). On the right hand side we have the magnitude of shear stress τ in Fig. 1(a). We therefore see that $\sigma = \tau$.

We now require that applying conventional stress gives rise to the same strain of the cube along the axis d_1. The strain must be

$$\epsilon = \frac{\sigma}{Y} + \nu\frac{\sigma}{Y}. \tag{6}$$

The first term is the usual strain and the second originates from the fact that we also push in the horizontal direction (and it is therefore positive). We can now set this stain equal to the one obtained for a shear strain (4)

$$\frac{\tau}{2G} = \frac{\sigma}{Y}(1+\nu). \tag{7}$$

Since we have also shown that $\sigma = \tau$, this gives the desired relation (1).