



**Figure 1:** (a) Situation for the calculation of the local electric field. We imagine a very large capacitor, filled with a dielectric. The spherical “cavity” in the centre is also filled with the dielectric and only serves as an artificial construction to split up the total electric field (at the red dot) into parts that can be easily calculated. (b) Construction for the calculation of the electric field at the centre of the cavity.

## The Local Field inside a Dielectric

In this note we derive the proper expression for the local field (9.7) in the book for the case of a uniform dielectric in a plane-plate capacitor. Consider the situation in Fig. 1(a) where we have a plane plate capacitor filled with a dielectric. We know the average electric field in the dielectric, the field we have called  $\mathcal{E}$  in the book.  $\mathcal{E}$  has two contributions

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1, \quad (1)$$

where  $\mathcal{E}_0 = \sigma/\epsilon_0$  is the contribution from the charges on the plate of the capacitor and  $\mathcal{E}_1 = -\mathbf{P}/\epsilon_0$  is the contribution from the charges on the surface of the dielectric. As discussed in the book,  $\mathcal{E}_1$  opposes  $\mathcal{E}_0$  and thus leads to a reduction of  $\mathcal{E}$ .

In order to calculate the local field in the middle of the dielectric, we now “artificially” introduce two additional fields. Imagine a little sphere in the dielectric that is small on the scale of the capacitor but large on an atomic scale. The sphere shall be filled with dielectric as well, it is only an artificial construction to help with the calculation because we can calculate the field from everywhere except the inside of the sphere macroscopically and then the field from the inside of the sphere microscopically. We can write the total field at the point in the centre of the sphere as

$$\mathcal{E}_{loc} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{E} + \mathcal{E}_2 + \mathcal{E}_3, \quad (2)$$

where  $\mathcal{E}_2$  and  $\mathcal{E}_3$  are the fields created by the charges on the surface of the sphere and by the microscopic inside of the sphere, respectively.

We can calculate  $\mathcal{E}_2$  as the field in the centre of a sphere with radius  $r$  and some surface charge  $\sigma$ . The situation is shown in Fig. 1(b). We find

$$\mathcal{E}_2 = \int \frac{1}{4\pi\epsilon_0} \frac{\sigma}{r^2} \hat{\mathbf{r}} dS. \quad (3)$$

Because of the symmetry of the problem, only the  $z$ -component of this is non-zero ( $z$  being the direction between the plates) and so we can simplify it as

$$\mathcal{E}_{2,z} = \int \frac{1}{4\pi\epsilon_0} \frac{\sigma}{r^2} \cos \Theta dS. \quad (4)$$

The surface charge density of the sphere is equivalent to the surface charge density of the dielectric at the planes of the capacitor (as the polarization is the same throughout the material) and we thus have that  $\sigma = P \cos \Theta$ . With this we find

$$\mathcal{E}_{2,z} = \int \frac{1}{4\pi\epsilon_0} \frac{P \cos \Theta}{r^2} \cos \Theta dS = \quad (5)$$

$$\int_0^{2\pi} \int_0^\pi \frac{1}{4\pi\epsilon_0} \frac{P \cos \Theta}{r^2} \cos \Theta r^2 \sin \Theta d\phi d\Theta = \frac{1}{3\epsilon_0} P. \quad (6)$$

The field  $\mathcal{E}_2$  clearly has the same direction as the external field  $\mathcal{E}_0$  and it is opposite to the field  $\mathcal{E}_1$ . If we assume for the moment that the microscopic  $\mathcal{E}_3 = 0$ , and use the general relation

$$\mathbf{P} = (\epsilon - 1)\epsilon_0 \mathcal{E}, \quad (7)$$

this already gives the desired relation for the local field as

$$\mathcal{E}_{loc} = \mathcal{E} + \frac{1}{3\epsilon_0} (\epsilon - 1)\epsilon_0 \mathcal{E} = \frac{1}{3} (\epsilon + 2) \mathcal{E}. \quad (8)$$

What we still have to show is that the microscopic field  $\mathcal{E}_3$  is zero. This is, in fact, not necessarily the case but it is so for many crystal structures. We show this for the cubic crystal structure. In this situation, the centre of the spherical cavity is placed on a lattice site in the cubic lattice and we have to sum the microscopic dipole moments due to all the dipoles in the cavity. The general electric field for one such dipole is

$$\mathcal{E}(\mathbf{r}) = \frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r} - r^2 \mathbf{p}}{4\pi\epsilon_0 r^5}. \quad (9)$$

In our specific case, all the dipoles will be formed in the  $z$  direction and so we are only interested in this direction. We thus get

$$\mathcal{E}_z(r) = p \frac{3z^2 - r^2}{4\pi\epsilon_0 r^5}, \quad (10)$$

leading to a total field at the centre of

$$\mathcal{E}_{3,z} = p \sum_i \frac{3z_i^2 - r_i^2}{4\pi\epsilon_0 r_i^5}, \quad (11)$$

where the summation is over all the dipoles in the cavity. Equivalently, we can write

$$\mathcal{E}_{3,z} = p \sum_i \frac{2z_i^2 - x_i^2 - y_i^2}{4\pi\epsilon_0 r_i^5}. \quad (12)$$

Now we can use the cubic symmetry of the lattice that gives us

$$\sum_i \frac{z_i^2}{r_i^5} = \sum_i \frac{x_i^2}{r_i^5} = \sum_i \frac{y_i^2}{r_i^5} \quad (13)$$

such the  $\mathcal{E}_{3,z}$  and  $\mathcal{E}_3$  are both zero.

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